The Loopless Walsh-Hadamard Operator

Let H be a Walsh-Hadamard matrix and let

$D = \mathrm{diag}(H)$.

Define

$A = H - D$.

The matrix A is obtained by removing all diagonal entries from H.

Basic Properties

The diagonal of A is identically zero:

$A_{ii} = 0$.

Consequently, for

$y = Ax$,

input coordinate x_i has **no direct effect** on output coordinate y_i:

Each output depends only on the remaining coordinates.

The trace is therefore

$\mathrm{tr}(A) = 0$.

Interpreting H as a signed graph adjacency matrix, A is the corresponding **loopless graph operator** obtained by deleting all self-loops.

Computing Ax

For the standard (unnormalized) Walsh-Hadamard transform:

$y = Hx$.

Then compute

$Ax = y - Dx$.

The diagonal entries of (D) are the Thue-Morse sequence

$D_{ii} = (-1)^{\text{popcount}(i)}$

Thus implementation requires:

1. One FWHT.
2. One elementwise signed subtraction.

Overall complexity remains

$O(n \log n)$.

Remarks

The operator (A=H-D):

- has zero diagonal,
- has zero trace,
- preserves the dense mixing structure of the Walsh-Hadamard transform,
- prevents any coordinate from directly contributing to its corresponding output coordinate,
- retains O(nlogn) evaluation cost.

It can therefore be viewed as a **loopless Walsh-Hadamard transform** or **loopless Walsh graph operator**.